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STRONG CONVERGENCE THEOREMS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACE

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ABSTRACT. We prove for a nonexpansive mapping T that under certain conditions the strong $\lim_{t\to 1^-} G_t(x)$ exists and is a fixed point of T, where $G_t(x) = (1-t)x + tTG_t(x)$, $0 \le t < 1$.

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|$$

for all x, y in C.

Let E^* be the dual space of E. Then the value of $f \in E^*$ at $x \in E$ will be denoted by $< x, f >$. With each $x \in E$, we associate the set

$$J(x) = \{f \in E^* :< x, f >= \|x\|^2 = \|f\|^2\}.$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \phi$ for each $x \in E$. The multivalued operator $J : E \to E^*$ is called the duality mapping of E. Let $B = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the norm of E is said to be Gateaux differentiable (and E is said to be smooth) if

$$\lim_{r\to 0} \frac{\|x + ry\| - \|x\|}{r}$$

exists for each x and y in B. It is said to be Frechet differentiable if for each x in B, this limit is attained uniformly for y in B. Finally, it is said to be uniformly Frechet differentiable (and E is said to be uniformly smooth) if the limit is attained uniformly for (x, y) in $B \times B$. It is well known that if E is smooth, then the duality mapping J is single valued. It is also known that if E has a Frechet differentiable norm, then J is norm to norm continuous.

The purpose of this note is to continue the discussion concerning the strong convergence of the path $t \to G_t(x)$, $0 \le t < 1$ defined by (1) below for each x in C. We prove for a nonexpansive mapping T that under certain conditions the strong $\lim_{t\to 1^-} G_t(x)$ exists and is a fixed point of T. The first results of this nature were established by Brower([2]) and Browder and Petryshyn([4]).

Key words and phrases. Nonexpansive mapping, Banach limit, Fixed points.

2. Lemmas

Let E be a Banach space. Then the modulus of convexity of E is defined as $\delta_E(\varepsilon) = \inf\{1 - \frac{1}{2}\|x+y\| : x, y \in B_E \text{ and } \|x-y\| \geq \varepsilon\}$, where $B_E = \{x \in E : \|x\| \leq 1\}$ is the closed unit ball of E. We recall that E is said to have the modulus of convexity of power type $p \geq 2$ (and E is said to be p-uniformly convex) if there exists a constant $c > 0$ such that

$$\delta_E(\varepsilon) \geq c\varepsilon^p$$

for $0 < \varepsilon \leq 2$.

We now define the mapping $G_t : C \to C$ by

$$G_t(x) = (1-t)x + tTG_t(x) \tag{1}$$

for all x in C and $0 \leq t < 1$. It is clear that for each $0 \leq t < 1$, the fixed point set of G_t coincides with that of T.

We also recall that a Banach limit LIM is a bounded linear functional on ℓ^∞ of norm 1 such that

$$\liminf_{n\to\infty} x_n \leq \mathrm{LIM}\{x_n\} \leq \limsup_{n\to\infty} x_n$$

and

$$\mathrm{LIM}\{x_n\} = \mathrm{LIM}\{x_{n+1}\}$$

for all $\{x_n\}$ in ℓ^∞.

LEMMA 1. *(Prus and Smarzewski [6]) Let E be a p-uniformly convex Banach space ($p > 1$). Then there exists a constant $c > 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - cW_p(\lambda)\|x-y\|^p \tag{2}$$

for all $x, y \in E$ and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

LEMMA 2. *Let C be a nonempty closed convex and bounded subset of a p-uniformly convex Banach space E, and let $\{x_n\}$ be a bounded sequence in E. We define the functional $r : C \to R$ by the formular*

$$r(x) = LIM\{\|x_n - x\|^p\}.$$

Then $r(\cdot)$ is continuous and convex.

Proof. For $x, y \in C$, we have

$$\big|\|x_n - x\|^p - \|x_n - y\|^p\big| \leq p(\mathrm{diam}C)^{p-1}\big|\|x_n - x\| - \|x_n - y\|\big|$$

and

$$\begin{aligned} |r(x) - r(y)| &= |\mathrm{LIM}\{\|x_n - x\|^p\} - \mathrm{LIM}\{\|x_n - y\|^p\}| \\ &\leq p(\mathrm{diam}C)^{p-1}\mathrm{LIM}\{|\|x_n - x\| - \|x_n - y\||\} \\ &\leq p(\mathrm{diam}C)^{p-1}\mathrm{LIM}\{\|x - y\|\} \\ &\leq p(\mathrm{diam}C)^{p-1}\|x - y\|. \end{aligned}$$

For any fixed $n \in N$ and $0 < t < 1$, by the inequality (2), we get

$$\begin{aligned}\|x_n - ((1-t)x + ty)\|^p &= \|(1-t)(x_n - x) + t(x_n - y)\|^p \\ &\le (1-t)\|x_n - x\|^p + t\|x_n - y\|^p - cW_p(t)\|x-y\|^p \\ &\le (1-t)\|x_n - x\|^p + t\|x_n - y\|^p.\end{aligned}$$

Taking the Banach limit LIM on each side, we obtain

$$\mathrm{LIM}\{\|x_n - ((1-t)x + ty)\|^p\} \le (1-t)\mathrm{LIM}\{\|x_n - x\|^p\} + t\mathrm{LIM}\{\|x_n - y\|^p\}.$$

Therefore we get

$$r((1-t)x + ty) \le (1-t)r(x) + tr(y).$$

LEMMA 3. *Let C be a nonempty closed convex subset of a p-uniformly convex and uniformly smooth Banach space E. Let $\{x_n\}$ be a bounded sequence in E. Then for $z_0 \in C$,*

$$\mathrm{LIM}\{\|x_n - z_0\|^p\} = \min_{y \in C} \mathrm{LIM}\{\|x_n - y\|^p\}$$

if and only if

$$\mathrm{LIM}\{< z - z_0, J(x_n - z_0) >\} \le 0$$

for all $z \in C$.

Proof. We first assume that $\mathrm{LIM}\{\|x_n - z_0\|^p\} = \min_{y\in C} \mathrm{LIM}\{\|x_n - y\|^p\}$. For $z \in C$ and $\lambda : 0 \le \lambda \le 1$, we have

$$\begin{aligned}\|x_n - z_0\|^p &= \|x_n - \lambda z_0 - (1-\lambda)z + (1-\lambda)(z - z_0)\|^p \\ &\ge \|x_n - \lambda z_0 - (1-\lambda)z\|^p \\ &\quad + p(1-\lambda) < z - z_0, J(x_n - \lambda z_0 - (1-\lambda)z) >\end{aligned}$$

since $J(x)$ is subdifferential of the convex function $\frac{1}{p}\|x\|^p$([3,p97]). Let $\varepsilon > 0$ be given. Since E is uniformly smooth, the duality map is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^* ([3]). Therefore,

$$| < z - z_0, J(x_n - \lambda z_0 - (1-\lambda)z) - J(x_n - z_0) > | < \varepsilon$$

if λ is close enough to 1. Consequently, we have

$$\begin{aligned}< z - z_0, J(x_n - z_0) > &< \varepsilon + < z - z_0, J(x_n - \lambda z_0 - (1-\lambda)z) > \\ &\le \varepsilon + \frac{1}{p(1-\lambda)}\{\|x_n - z_0\|^p \\ &\quad - \|x_n - \lambda z_0 - (1-\lambda)z\|^p\}.\end{aligned}$$

Therefore, we have

$$\mathrm{LIM}\{< z - z_0, J(x_n - z_0) >\} \le 0$$

for all $z \in C$.

To prove reverse, let $z \in C$. Then since

$$\|x_n - z\|^p - \|x_n - z_0\|^p \ge p < z_0 - z, J(x_n - z_0) >$$

for all $n \in N$ and $\mathrm{LIM}\{< z - z_0, J(x_n - z_0) >\} \le 0$ for all $z \in C$, we have

$$\mathrm{LIM}\{\|x_n - z_0\|^p\} = \min_{z\in C} \mathrm{LIM}\{\|x_n - z\|^p\}.$$

LEMMA 4. *Let C be a closed convex and bounded subset of a p-uniformly convex and uniformly smooth Banach space E, and $\{x_n\}$ be a bounded sequence of E. Then, the set*

$$M = \{u \in C : LIM\{\|x_n - u\|^p\} = \min_{z \in C} LIM\{\|x_n - z\|^p\}\}$$

consists of one point.

Proof. Let $g(z) = \mathrm{LIM}\{\|x_n - z\|^p\}$ for every $z \in C$ and $r = \inf\{g(z) : z \in C\}$. Then, by Lemma 2, the function g on C is convex and continuous and $g(z) \to \infty$ as $\|z\| \to \infty$. From [1,p79], there exists $u \in C$ with $g(u) = r$. Therefore M is nonempty. By Lemma 3, we know that $u \in M$ if and only if $\mathrm{LIM}\{< z - u, J(x_n - u) >\} \leq 0$ for all $z \in C$. We show that M consists of one point. Let $u, v \in M$ and suppose $u \neq v$. Then by [7,Theorem 1], there exists a positive number ε such that

$$< x_n - u - (x_n - v), J(x_n - u) - J(x_n - v) > \geq \varepsilon$$

for every $n \in N$. Therefore

$$\mathrm{LIM}\{< v - u, J(x_n - u) - J(x_n - v) >\} \geq \varepsilon > 0.$$

On the other hand, since $u, v \in M$, we have

$$\mathrm{LIM}\{< u - v, J(x_n - v) >\} < 0$$

and

$$\mathrm{LIM}\{< v - u, J(x_n - u) >\} < 0.$$

Then we have

$$\mathrm{LIM}\{< v - u, J(x_n - u) - J(x_n - v) >\} < 0.$$

This is a contradiction. Therefore $u = v$.

3. Main Results

THEOREM 1. *Let C be a closed convex and bounded subset of a p-uniformly convex and uniformly smooth Banach space E, and $\{x_n\}$ be a bounded sequence of E such that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. If $T : C \to C$ is a nonexpansive mapping, then*

$$M = \{u \in C : LIM\{\|x_n - u\|^p\} = \min_{z \in C} LIM\{\|x_n - z\|^p\}\}$$

is a fixed point set of T.

Proof. We will show that the set M is invariant under T. In fact, since $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$, we have, for $u \in M$,

$$\begin{aligned} \mathrm{LIM}\{\|x_n - Tu\|^p\} &= \mathrm{LIM}\{\|Tx_n - Tu\|^p\} \\ &\leq \mathrm{LIM}\{\|x_n - u\|^p\} \end{aligned}$$

and hence $Tu \in M$. On the other hand, by Lemma 4, we know that M consists of one point. Therefore this point is a fixed point of T and M is a fixed point set of T.

It is well known in ([8]) that a uniform smooth space has normal structure. Since such a space is also reflexive, each bounded closed convex subset of it has the fixed point property for nonexpansive mappings ([5]).

THEOREM 2. *Let C be a closed convex and bounded subset of a p-uniformly convex and uniformly smooth Banach space E, $T : C \to C$ a nonexpansive mapping, and $G_t : C \to C$, $0 < t < 1$, the family of mappings defined by (1). Then, for each x in C, the strong $\lim_{t \to 1^-} G_t(x)$ exists and is a fixed point of T.*

Proof. Note that from the preceding statement T has a fixed point in C. Let w be a fixed point of T. Fix a point x in C, denote $G_t(x)$ by $y(t)$. Since $y(t) - w = (1-t)(x - w) + t(Ty(t) - Tw)$,

$$\|y(t) - w\| \le \|x - w\|$$

and $\{y(t)\}$ remains bounded as $t \to 1^-$. We also have

$$\begin{aligned} \lim_{t \to 1^-} \|y(t) - Ty(t)\| &= \lim_{t \to 1^-} \|(1-t)x - (1-t)Ty(t)\| \\ &= 0. \end{aligned}$$

Now let $t_n \to 1^-$ and $y_n = y(t_n)$. Define $f : C \to [0, \infty)$ by $f(z) = \mathrm{LIM}\{\|y_n - z\|^p\}$. From Lemma 2 f is continuous and convex, $f(z) \to \infty$ as $\|z\| \to \infty$, which implies that f attains its infimun over C. That is, there exists a $z_0 \in C$ such that

$$\mathrm{LIM}\{\|y_n - z_0\|^p\} = \min_{y \in C} \mathrm{LIM}\{\|y_n - y\|^p\}.$$

Let M be the set of minimizers of T. By Theorem 1, $z_0 \in M$ is the fixed point of T. Therefore

$$\begin{aligned} < y_n - Ty_n, J(y_n - z_0) > &=< y_n - Tz_0 + Tz_0 - Ty_n, J(y_n - z_0) > \\ &= \|y_n - Tz_0\|^2 - < Ty_n - Tz_0, J(y_n - z_0) > \\ &\ge \|y_n - Tz_0\|^2 - \|Ty_n - Tz_0\| \|y_n - z_0\| \\ &\ge \|y_n - Tz_0\|^2 - \|y_n - Tz_0\|^2 = 0 \end{aligned}$$

for all n. It follows that for $x \in C$,

$$\begin{aligned} 0 &\le < y_n - Ty_n, J(y_n - z_0) > \\ &=< (1-t_n)x + t_n Ty_n - Ty_n, J(y_n - z_0) > \\ &=< (1-t_n)x - (1-t_n)Ty_n, J(y_n - z_0) > \\ &= (1-t_n) < x - Ty_n, J(y_n - z_0) > \end{aligned}$$

for all n. Thus, we get for $x \in C$,

$$< y_n - x, J(y_n - z_0) > \le 0 \tag{3}$$

for all n. From Lemma 3

$$\mathrm{LIM}\{< z - z_0, J(y_n - z_0) >\} \le 0 \tag{4}$$

for all $z \in C$. Choosing $z = y_n$ in (4), we conclude that

$$\mathrm{LIM}\{\|y_n - z_0\|\} \leq 0.$$

Thus there is a subsequence of $\{y_n\}$ which converges strongly to z_0. To complete the proof, suppose that $y_{n_k} \to z_1$ and $y_{m_k} \to z_2$. Then by (3),

$$< z_1 - x, J(z_1 - z_2) > \leq 0$$

and

$$< z_1 - z_2, J(z_1 - z_2) > \leq 0.$$

Hence $z_1 = z_2$ and the strong $\lim_{t \to 1^-} y(t)$ exists, which completes the proof.

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